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The scattering of charged particles on the toroidal solenoid

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Abstract. The charged particle scattering amplitude on the magnetic field of a toroidal solenoid is obtained in the first Born and high-energy approximations. It is shown that multiconnectedness of the accessible space, non-triviality vector potentials in it and the single-valuedness of the wavefunctions used are not enough for the existence of the Aharonov–Bohm effect. Criteria are given for this. Surrounding the toroidal solenoid with barriers of different geometrical forms, we observe that the magnetic field contribution to the scattering amplitude depends crucially both on the barrier height and its form. The latter could hardly be explained by particle penetration into the $H \neq 0$ region. We propose an addendum to Tonomura's experiments which probably could clear up any doubts about the existence of the AB effect. The gedanken experiment is discussed which shows that the effect of inaccessible fields could be observed even in a simply connected space.

1. Introduction

In classical mechanics, the force acting on the charged particle is completely determined by the field strengths E, H: $\boldsymbol{F} = e(\boldsymbol{E} + \frac{1}{c}[v, \boldsymbol{H}])$. This means that there is no scattering on space regions with $\boldsymbol{E} = \boldsymbol{H} = 0$. The electric scalar potential φ and magnetic vector potential $\boldsymbol{A}$ play an auxiliary role in classical electromagnetic theory. They mainly serve to simplify the field equations. The gauge transformations of these potentials do not change the field strengths and, as a consequence, the particle equations of motion. So, in classical mechanics, the fundamental quantities are the field strengths $\boldsymbol{E}, \boldsymbol{H}$.

A quite different situation arises in quantum mechanics. The reason is that electromagnetic potentials, not field strengths, enter into the Schrödinger equation. The exceptional role of the electromagnetic potentials in quantum theory was discovered by Ehrenberg and Siday in 1949 [1]. Ten years later it was investigated in greater detail by Aharonov and Bohm [2]. Among other problems they considered the scattering of charged particles on the magnetic field of the infinite cylindrical solenoid. The scattering takes place even if the space regions with $\boldsymbol{E}, \boldsymbol{H} \neq 0$ are absolutely inaccessible for the incoming particles. This phenomenon, i.e. the influence of the inaccessible fields, is called the Aharonov–Bohm (AB) effect. Aharonov and Bohm used the single-valued wavefunctions. On the other hand, in multiconnected space (such as the exterior of the infinite cylindrical solenoid) there are possible non-equivalent representations of the angular momentum to which there correspond non-single-valued wavefunctions (see, e.g., [3–5] and the paper by Ohnuki in [6]). The equal status of these representations has given rise to the recent excited discussion on the existence of the AB effect (see, e.g., [7] and papers by Berry *et al* in [8]). Note that the AB effect arises only for single-valued wavefunctions [3-5, 7, 9]. On the other hand, the experiments performed on the cylindrical solenoid [10] and the toroidal one (see, e.g., the paper by Tonomura *et al* in [6]) clearly show that the diffraction pattern shifts when the current in the solenoid is switched on. In the so-called alternative (or hydrodynamic) interpretation of the AB effect the positive outcomes of these experiments are attributed to the penetration of the incoming particles into the space regions with $H \neq 0$ [11]. There are at least two reasons for this. The finite length of the real cylindrical solenoid results in the magnetic field leakages near the solenoid's ends. This permits one to interpret the above experiments as scattering either on the tails of the magnetic field or on the return flux [12, 13]. The second reason is due to the finite value of the real potential barrier which keeps particles out of the region with $H \neq 0$. An additional complication for the infinite cylindrical solenoid is due to the fact that the slow decreasing ($\sim r^{-1}$) of the vector potential modifies the incoming wavefunction which turns out not to be single valued. This leads to numerous paradoxes with the non-conservation of angular momentum [14]. These two drawbacks of the infinite cylindrical solenoid (i.e. the magnetic field tails and bad asymptotics of the incoming wavefunction) are lacking for the toroidal solenoid. This permits us to use plane-wave asymptotics for the incoming wavefunction and eliminates completely the above-mentioned paradoxes, so there are good reasons for treating the toroidal solenoid.

The plan of this paper is as follows. In § 2 we present the main facts concerning the toroidal solenoid. We define a function, a gradient of which is just the vector potential outside the solenoid. This function plays a fundamental role for the description of the scattering process. In view of its unfamiliarity we apply it first in § 3 to the well known case of a thin cylindrical solenoid. In §§ 4 and 5 we obtain the magnetic field contribution to the total scattering amplitude (call it magnetic scattering amplitude for short) in the first Born and high-energy approximations. In § 6 the formalism of the generating function is applied to the scattering in ideal multiconnected space. We discuss what are the conditions under which the magnetic field gives no contribution to the scattering amplitude. We present concrete examples showing that in the same ideal multiconnected space with non-zero vector potentials in it and with the single-valued wavefunctions the magnetic scattering amplitude may or may not vanish. In § 7 we discuss the conditions imposed on the wavefunction by the gauge transformation eliminating the vector potential outside the solenoid. In § 8 we surround the toroidal solenoid with the finite potential barriers of different geometries. It turns out that for some geometries the magnetic scattering amplitude remains finite as the barrier's height grows, while for others it vanishes. This fact could hardly be reconciled with the above-mentioned alternative interpretation of the AB effect experiments. In § 9 we present an addendum to Tonomura's experiments which probably may clear up any doubts about the existence of the AB effect. Finally, in § 10 we ask the following question: are the effects of the hidden (inaccessible) field completely unobservable in a simply connected space? We present a gedanken experiment showing that these effects could indeed be observable.

2. Some facts about the toroidal solenoid

Let the solenoid winding be performed on the torus $(\rho - d)^2 + z^2 = R^2$ whose axis coincides with the z one. The strength of the magnetic field equals zero outside the solenoid and

$$H_\rho = H_z = 0 \qquad H_\varphi = \frac{\phi}{2\pi\rho}[d-(d^2-R^2)^{1/2}]^{-1}$$

inside it. Here ϕ is the magnetic field flux through the solenoid cross section. In [15] the magnetic vector potential for such a solenoid was obtained in a Coulomb gauge (div $\boldsymbol{A}=0$). Its components are everywhere finite, continuous and single-valued functions of the coordinates. For large distances it falls as r^{-3}:

$$A_z \approx \frac{\phi}{16}\frac{R^2 d}{d-(d^2-R^2)^{1/2}}\frac{1+3\cos 2\theta}{r^3} \qquad A_\rho \approx \frac{3\phi}{16}\frac{R^2 d}{d-(d^2-R^2)^{1/2}}\frac{\sin 2\theta}{r^3}. \tag{2.1}$$

Here A_ρ and A_z are cylindrical components of A. Due to axial symmetry they do not depend on the azimuthal angle φ and $A_\varphi=0$; r and θ entering into (2.1) are the radial distance and the polar angle.

As outside the solenoid the vector potential is curlless, it can be presented as a gradient of some function χ [9, 16]. For convenience, we call this function the generating one. For closed contours passing through the solenoid hole $\oint A_l\,\mathrm{d}l \neq 0$. This suggests that χ is the discontinuous function. For the toroidal solenoid, it was calculated explicitly in [17]. The following two features of χ will be needed later. First, χ falls as r^{-2} at infinity:

$$\chi \approx -\frac{\phi\, dR^2}{8}\frac{\cos\theta}{r^2}[d-(d^2-R^2)^{1/2}]^{-1}. \tag{2.2}$$

Second, χ suffers a finite jump equal to $-\phi$ when one crosses a part $\rho \leqslant d-R$ of the equatorial $z=0$ solenoid plane. In other words: for the toroidal solenoid the discontinuities of the generating function fill the equatorial circle of the radius $d-R$.

The formalism of the generating function plays a major role in the derivation of the scattering amplitude. In view of its unfamiliarity and for the pedagogical purposes we demonstrate its effectiveness on the well known example of a thin cylindrical solenoid.

3. A pedagogical example: thin cylindrical solenoid

For the infinitely thin cylindrical solenoid with its axis directed along the z one the generating function equals $\chi=\phi\varphi/2\pi$. Hence it follows that $A_\rho=A_z=0$, $A_\varphi=\phi/2\pi\rho$ as it should be. We observe that discontinuities of the generating function fill the positive x semiaxis. In the first Born approximation† one obtains for the magnetic scattering amplitude:

$$f(\varphi)=\frac{e}{\hbar c}\frac{1}{(2\pi \mathrm{i}k)^{1/2}}\int \exp[-\mathrm{i}k(x'\cos\varphi+y'\sin\varphi)]\boldsymbol{A}\boldsymbol{\nabla}\psi_0\,\mathrm{d}x'\,\mathrm{d}y'. \tag{3.1}$$

Here ψ_0 is the wavefunction in the absence of the magnetic field (i.e. $\psi_0=\exp(\mathrm{i}kx)$).

† We know [18] that the difference of the first Born approximation for the infinite solenoid from the exact results in the limit of small magnetic fluxes is due not to the drawbacks of the Born approximation but rather to different physical situations.

Setting $A = \mathrm{grad}\,\chi$ and integrating twice in parts, we transform (3.1) to

$$f = \frac{e}{2\hbar c}\frac{1}{(2\pi \mathrm{i}k)^{1/2}}\int \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{div}\{\exp[-\mathrm{i}k(x'\cos\varphi + y'\sin\varphi)]\,\mathrm{grad}\,\chi\psi_0$$
$$-\chi\psi_0\,\mathrm{grad}\exp[-\mathrm{i}k(x'\cos\varphi+y'\sin\varphi)]\}.$$

Applying the Gauss theorem (with a modification due to the discontinuity of the χ function) this integral may be reduced to the linear integral over the circumference of the sufficiently large radius R_0

$$I_1 = \frac{1}{2}\frac{\gamma \mathrm{i}kR_0}{(2\pi \mathrm{i}k)^{1/2}}\int_0^{2\pi}\exp\{\mathrm{i}kR_0[\cos\varphi' + \cos(\varphi-\varphi')]\}$$
$$\times\varphi'[\cos\varphi' + \cos(\varphi-\varphi')]\,\mathrm{d}\varphi' \tag{3.2}$$

and that over the discontinuities of the χ function

$$I_2 = \frac{\pi \mathrm{i}k\gamma}{(2\pi \mathrm{i}k)^{1/2}}\sin\varphi\int_0^{R_0}\mathrm{d}x\,\exp[\mathrm{i}kx(1-\cos\varphi)]$$
$$= \frac{\pi\gamma}{(2\pi \mathrm{i}k)^{1/2}}\frac{\sin\varphi}{1-\cos\varphi}\{\exp[\mathrm{i}kR_0(1-\cos\varphi)]-1\}.$$

Taking into account that the integral in (3.2) rapidly oscillates for $R_0 \to \infty$ and using the steepest descent method one obtains:

$$I_1 = -\frac{\pi\gamma}{(2\pi \mathrm{i}k)^{1/2}}\frac{\sin\varphi}{1-\cos\varphi}\exp[\mathrm{i}kR_0(1-\cos\varphi)] \qquad \gamma = \frac{e\phi}{hc}.$$

Adding I_1 and I_2 results in

$$f_{AB} = -\frac{\pi\gamma}{(2\pi \mathrm{i}k)^{1/2}}\frac{\sin\varphi}{1-\cos\varphi}$$

i.e. one recovers the usual AB amplitude for the point solenoid.

4. The perturbative scattering amplitude on the magnetic field of the toroidal solenoid

Here we consider the scattering of the charged zero-spin particles on the curlless vector potential of the toroidal solenoid. In order to keep incoming particles out of the region with $H \neq 0$, one screens this field using the infinite repulsive potential of the suitable geometrical form. The repulsive potential being infinitely high, the wavefunction vanishes on the boundary of the impenetrable screen (as well as inside it). So, one should solve the Schrödinger equation

$$-\frac{\hbar^2}{2\mu}\left(\nabla - \frac{\mathrm{i}e}{\hbar c}A\right)^2\psi + V\psi = E\psi \tag{4.1}$$

where $V = \infty$, $\psi = 0$ inside the screen and at its boundary and $V = 0$ otherwise. Consider first the case when the solenoid T_0 is embedded into a screen of the toroidal form. For simplicity, we take it to coincide with T_0. We restrict ourselves to the first-order perturbation theory WRT the vector potential A. Then

$$\psi = \psi_0 + \frac{2\mathrm{i}e}{\hbar c}\int G_0(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{A}\cdot\boldsymbol{\nabla}\psi_0\,\mathrm{d}V'. \tag{4.2}$$

Here ψ_0 and G_0 are the wave and Green functions, respectively, corresponding to the scattering on the impenetrable torus in the absence of the magnetic field. They vanish on the surface of T_0. Integration in (4.2) runs outside the solenoid where $A = \text{grad}\,\chi$. Further, one has the identity

$$A\,\text{grad}\,\psi_0 = \text{grad}\,\chi\,\text{grad}\,\psi_0 = \tfrac{1}{2}\Delta(\chi\psi_0) - \tfrac{1}{2}\chi\,\Delta\psi_0 = \tfrac{1}{2}(\Delta + k^2)\psi_0\chi$$

(Δ and κ are the Laplace operator and wavenumber respectively). Then

$$\psi = \psi_0 + \frac{\mathrm{i}e}{\hbar c}\int G_0(\Delta + k^2)\chi\psi_0\,\mathrm{d}V'. \tag{4.3}$$

Integrating (4.3) twice by parts and using the equation for G_0 one obtains:

$$\psi = \psi_0 + \frac{\mathrm{i}e}{\hbar c}\chi\psi_0 + \frac{\mathrm{i}e}{\hbar c}\int \text{div}(G_0\,\text{grad}\,\chi\psi_0 - \chi\psi_0\,\text{grad}\,G_0)\,\mathrm{d}V'. \tag{4.4}$$

To calculate the scattering amplitude, we need only the limit of (4.4) as $r \to \infty$. The first term in the RHS of (4.4) gives the incident plane wave (due to the vector potential asymptotic behaviour (2.1)) and the scattering amplitude on the impenetrable torus in the absence of the magnetic field. The second term may be neglected as χ goes to zero as r^{-2} for $r \to \infty$. Now consider the third term in (4.4). Forget for the moment that the vector function on which the divergence operator acts is discontinuous (due to the χ function). Applying the Gauss theorem one can replace the volume integration by that over the surface enveloping this volume. This surface consists of two parts: the surface of the torus T_0 and that of the infinite radius sphere C_R. The integral over the torus surface vanishes since both ψ_0 and G_0 equal zero there. So, there remains the integral over C_R

$$R^2\int\left(G_0\frac{\partial\chi\psi_0}{\partial R} - \chi\psi_0\frac{\partial G_0}{\partial R}\right)\mathrm{d}\Omega' \qquad (\mathrm{d}\Omega' = \sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi').$$

Due to the asymptotic behaviour of the χ function and to the rapid oscillations of the integrand, the above integral tends to zero like R^{-2} as $R \to \infty$ [17]. Now we take into account the discontinuous nature of the χ function. A careful examination (see [17] or [19]) shows that the Gauss theorem should be modified: the following integral over the discontinuity region should be added to the mentioned surface integrals

$$\frac{\mathrm{i}e\phi}{\hbar c}\int_{\rho' \leqslant d-R,\, z'=0}\left(G_0\frac{\partial\psi_0}{\partial z'} - \psi_0\frac{\partial G_0}{\partial z'}\right)\rho'\,\mathrm{d}\rho'\,\mathrm{d}\varphi'. \tag{4.5}$$

The wave equation is not separated in toroidal coordinates, so one should find some substitute for ψ_0 and G_0 in (4.5). Following the Kirchhoff method in optics [20], we approximate them on the solenoid hole by the unperturbed values

$$\psi_0 \approx \exp(\mathrm{i}\boldsymbol{kr}) \qquad G_0 \approx -\frac{1}{4\pi}\frac{\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{4.6}$$

Finally, passing to the limit $r \to \infty$ in (4.5), one obtains the magnetic scattering amplitude (the initial wavevector is directed along the z axis):

$$\begin{aligned} f_1 &= \frac{ke\phi}{4\pi\hbar c}(1+\cos\theta)\int \exp(-\mathrm{i}k\rho'\cos\varphi\sin\theta)\rho'\,\mathrm{d}\rho'\,\mathrm{d}\varphi \\ &= \frac{e\phi}{2\hbar c}(d-R)\frac{1+\cos\theta}{\sin\theta}J_1[k(d-R)\sin\theta]. \end{aligned} \tag{4.7}$$

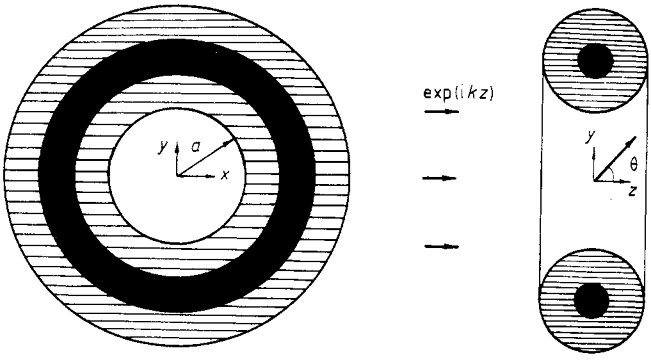

Figure 1. The toroidal solenoid (dark) is embedded into the toroidal potential barrier of height V_0. The magnetic scattering amplitude tends to a finite value (4.9) as $V_0 \to \infty$.

For the thin solenoid ($R/d \ll 1$) this results in

$$\frac{e\phi d}{2\hbar c}\frac{1+\cos\theta}{\sin\theta}J_1(kd\sin\theta). \qquad (4.8)$$

Equation (4.8) was obtained earlier in a very interesting paper [21] but the procedure used there gives rise to doubt. In fact, the vector potentials used in it had δ-type singularities on the $z=0$ plane. As a result, the neglected term

$$-\frac{e^2}{\hbar^2 c^2}\boldsymbol{A}^2$$

turns out to be more singular than taking into account

$$-\frac{ie}{\hbar c}(2\boldsymbol{A}\nabla\psi_0+\psi_0\,\mathrm{div}\,\boldsymbol{A})\dagger.$$

The integral

$$\iiint \exp(-ik\boldsymbol{n}\boldsymbol{r}')A^2\psi_0\,\mathrm{d}V'$$

diverges and this leads to the infinite scattering amplitude. On the contrary, as the potentials used here are everywhere continuous, finite and single-valued functions, the terms quadratic in the potential may obviously be neglected for small values $e\phi/\hbar c$.

Equation (4.7) is easily generalised for the impenetrable toroidal screen which encompasses the solenoid T_0 (see figure 1):

$$f_1=\frac{e\phi a}{2\hbar c}\frac{1+\cos\theta}{\sin\theta}J_1(ka\sin\theta). \qquad (4.9)$$

The parameter a is shown in the same figure.

5. High-energy scattering amplitude

Now we calculate the scattering amplitude on the toroidal solenoid in the high-energy

† div $\boldsymbol{A}\neq 0$ for the vector potential used in [21].

approximation. For this we make in the Lippmann–Schwinger equation

$$\psi = \psi_0 + \int G_0(\boldsymbol{r}, \boldsymbol{r}') V_1 \psi(\boldsymbol{r}')\, \mathrm{d}V' \qquad \left(V_1 = \frac{2\mathrm{i}e}{\hbar c} \boldsymbol{A}\nabla + \frac{e^2}{\hbar^2 c^2} \boldsymbol{A}^2 \right)$$

the following simplifications [22], typical for this method. Change the Green function G_0 by the plane function;

$$-\frac{1}{4\pi} \exp(\mathrm{i}k|\boldsymbol{r} - \boldsymbol{r}'|)|\boldsymbol{r} - \boldsymbol{r}'|^{-1}.$$

Instead of ψ make use under the integral of its high-energy approximation:

$$\exp\left(\mathrm{i}kz + \int_{-\infty}^{z} A_z \, \mathrm{d}z \right).$$

This is correct only if $e/(\hbar ck)\boldsymbol{A}^2 \ll |A_z|$. Otherwise, the terms quadratic in the vector potential should be included in the integrand. This in turn leads to the infinite scattering amplitude for the singular vector potentials used in [21]. The potentials used in the present work are everywhere finite and continuous. Thus, the above condition is fulfilled and one has for the scattering amplitude:

$$f_1(\boldsymbol{n}) = \frac{ek}{2\pi\hbar c} \int \exp(\mathrm{i}\boldsymbol{q}\boldsymbol{r}') \cdot A_z \exp\left(\frac{\mathrm{i}e}{\hbar c} \int_{-\infty}^{z'} A_z \, \mathrm{d}z \right) \mathrm{d}V'.$$

Here $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$ is the momentum transfer; $\boldsymbol{k}' = \boldsymbol{n}k$, $\boldsymbol{k} = \boldsymbol{e}_z k$. As for high energies the small-angle scattering dominates, the vector $\boldsymbol{q}$ may be considered to be perpendicular to the initial wavevector $\boldsymbol{k}$; therefore $\boldsymbol{q}$ lies in the $z = 0$ plane. Then

$$\begin{aligned} f_1(\boldsymbol{n}) &= \frac{ek}{2\pi\hbar c} \int \mathrm{d}^2\boldsymbol{\rho} \exp(\mathrm{i}\boldsymbol{q}\boldsymbol{\rho}) \int_{-\infty}^{z} \mathrm{d}z A_z \exp\left(\frac{\mathrm{i}e}{\hbar c} \int_{-\infty}^{z} A_z \, \mathrm{d}z' \right) \\ &= -\mathrm{i}k(d - R) \left[\exp\left(\frac{\mathrm{i}e\phi}{\hbar c} \right) - 1 \right] J_1[q(d - R)]/q. \end{aligned}$$

As expected, this high-energy amplitude coincides with the perturbative one (4.7) for small values of $e\phi/\hbar c$ and angles θ.

The following generalisation for the case presented in figure 1 is evident:

$$f_1(\boldsymbol{n}) = -\mathrm{i}ka \left[\exp\left(\frac{\mathrm{i}e\phi}{\hbar c} \right) - 1 \right] J_1(qa)/q.$$

6. The scattering in ideal multiconnected spaces

In the previous two sections we have considered the scattering of charged particles on the toroidal solenoid shielded with the impenetrable torus which either coincides with the solenoid or encloses this solenoid in a manner presented in figure 1. Now we surround the toroidal solenoid with different impenetrable screens and study particle scattering on the accessible magnetic fields. Some digression is needed however. We note that outside the solenoid the Schrödinger equation (4.1) is satisfied by the following expression:

$$\psi = \psi_0 \exp\left(\frac{\mathrm{i}e\chi}{\hbar c} \right) \tag{6.1}$$

where χ is the generating function defined in § 2 and ψ_0 is the wavefunction in the absence of the magnetic field. The transformation (6.1) is unitary and this guarantees the coincidence of all the observables for ψ and ψ_0. We always choose ψ (which is a solution of the unabridged Schrödinger equation (4.1)) to be single valued. So, in general (due to the discontinuity of the χ function) ψ_0 in (6.1) is the multivalued wavefunction. It may happen that discontinuities of the χ function lie completely inside the inaccessible region. There $\psi = \psi_0 = 0$ and (6.1) is a unitary transformation between two single-valued functions with (ψ) and without (ψ_0) magnetic field. In this case a magnetic field does not lead to the observable effects. Hence, there is no room for the AB effect.

After this digression, we consider concrete examples. In figure 2 the toroidal solenoid is embedded into the impenetrable infinite cylinder C_0. As the singularities of the toroidal generating function lie all inside C_0 (where $\psi = 0$), there is no AB effect.

Another example of the multiconnected topological space with non-zero vector potential in it is shown in figure 3 where the toroidal solenoid is embedded into one

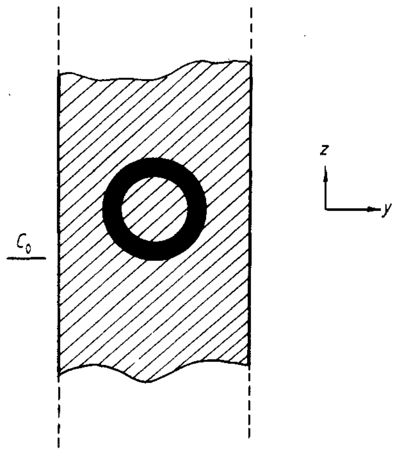

Figure 2. The toroidal solenoid is embedded into the infinite cylindrical potential barrier of height V_0. The magnetic scattering amplitude tends to zero as $V_0 \to \infty$.

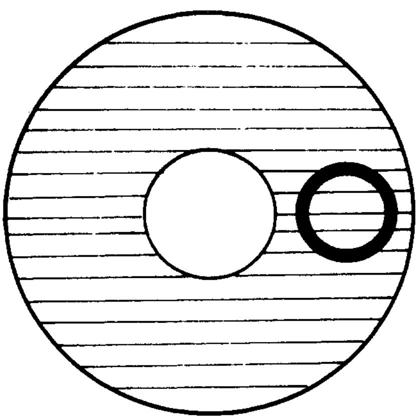

Figure 3. The toroidal solenoid is embedded into one arm of the toroidal potential barrier of height V_0. The magnetic scattering amplitude tends to zero as $V_0 \to \infty$.

'arm' of the impenetrable torus. As in the previous case, there is no AB scattering. This means that for the same multiconnected space (i.e. the exterior of the torus) the existence of the AB effect depends crucially on the way in which the toroidal solenoid is embedded into this torus (there is an AB effect for figure 1 and no such effect for figure 3).

We conclude that the existence of the AB effect is not determined solely by the multiconnectedness of the accessible space, non-triviality of the vector potentials in it or the single-valuedness of the wavefunction. The condition for the existence of the AB effect may be expressed either in terms of the generating function (it exists if the discontinuities of this function lie in the accessible region) or non-integrable phase factor [23] (there are closed, accessible for particles, paths along which $\oint A_l \, \mathrm{d}l \neq 0$).

7. On the gauge transformations which change vector potentials outside the solenoid

The fact that $\boldsymbol{A} = \operatorname{grad} \chi$ outisde the solenoid raises the hope of eliminating the vector potential there via the suitable gauge transformation. It has been shown [9, 24] that for one cylindrical solenoid it leads to the singular magnetic field on the solenoid axis directed opposite to the initial magnetic field. For the toroidal solenoid wound on the torus $(\rho - d)^2 + z^2 = R^2$, the gauge transformation eliminating the vector potential outside the solenoid gives rise to the singular magnetic field

$$H = -\frac{\phi \delta(z) \delta(d - R)}{2\pi}.$$

As these gauge transformations are singular, they also modify the boundary properties of the wavefunctions: being single-valued before the gauge transformation, they become multivalued after it. The boundary properties of the wavefunctions are unchanged if one uses a non-singular gauge transformation. As an example, consider the infinite cylindrical solenoid. The strength of the magnetic field is constant inside the solenoid ($H_x = H_y = 0$, $H_z = H$) and zero outside it. In addition to the usually used vector potential ($A_x = -HR^2 y/2\rho^2$, $A_y = HR^2 x/2\rho^2$ outside the solenoid and $A_x = -Hy$, $A_y = Hx$ inside it), one may equally work with the following one (figure 4): $A'_x = 0$ everywhere, $A'_y = H[x + (R^2 - y^2)^{1/2}]$ inside the solenoid. Outside it A'_y differs from zero inside the shaded strip $-R < y < R, x > 0$: $A'_y = 2H(R^2 - y^2)^{1/2}$. These vector potentials are connected through the non-singular gauge transformation: $\boldsymbol{A} =$

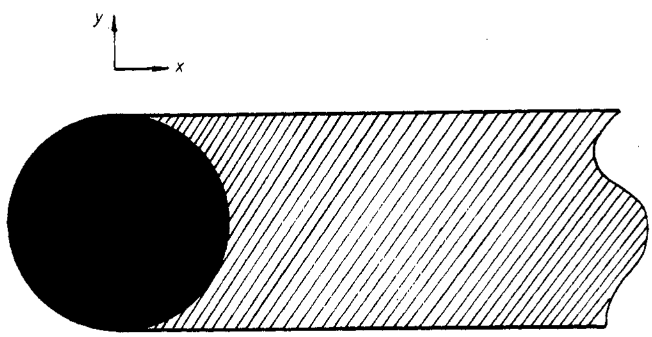

Figure 4. The vector potential of the cylindrical solenoid in a non-standard gauge. Outside the solenoid $\boldsymbol{A}$ differs from zero in the shaded region only.

$\mathbf{A}' - \operatorname{grad} \partial\alpha/\partial y$, where α satisfies the equation

$$\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}=Hf(x, y).$$

The function f equals $x+(R^2-y^2)^{1/2}$ inside the solenoid. Outside the solenoid it equals $2(R^2-y^2)^{1/2}$ in the shaded region and zero otherwise. For the infinitely thin solenoid $\mathbf{A}'$ degenerates into

$$A'_x=0 \qquad A'_y=\phi\delta(y)\theta(x) \qquad \phi=\pi R^2 H$$

i.e. in this gauge the vector potential is everywhere zero except for the positive x semiaxis.

Earlier, evaluating the scattering amplitude on the toroidal solenoid $(\rho-d)^2+z^2=R^2$ we have used the vector potential which falls as r^{-3} at large distances (see (2.1)). In [15] the vector potential was obtained for which the single non-vanishing component A_z differs from zero only in the nearest vicinity of the solenoid. It equals

$$g \ln \frac{d+(R^2-z^2)^{1/2}}{\rho}$$

inside the solenoid and

$$g \ln \frac{d+(R^2-z^2)^{1/2}}{d-(R^2-z^2)^{1/2}}$$

in a shaded region (see figure 5). Here

$$g=\frac{\phi}{2\pi[d-(d^2-R^2)^{1/2}]^{-1}}$$

and ϕ is the magnetic flux. In [15] the gauge transformation between these two vector potentials was found.

Here is a conclusion. We may use a gauge transformation eliminating the vector potential in some regions of space. The single-valued wavefunctions may be used if the function-generating gauge transformation is well behaved. Otherwise, care should be paid to the boundary conditions of the wavefunctions. These considerations are

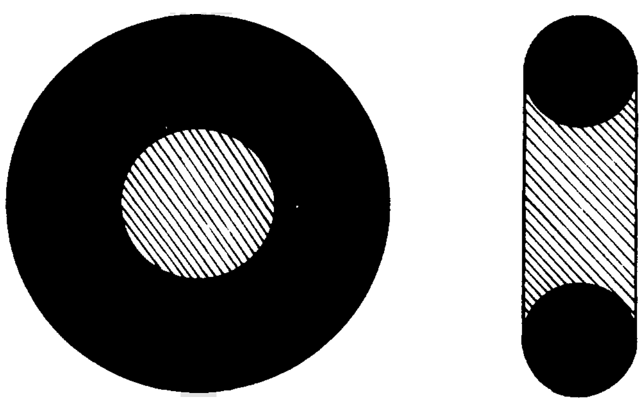

Figure 5. The same as in the previous figure but for the toroidal solenoid.

rather trivial but disregarding them has given rise to numerous paradoxical and erroneous conclusions.

8. The scattering on the magnetic field for the potential barriers which are not completely impenetrable

We return again to (6.1). In general it connects the single-valued and non-single-valued wavefunctions. In the previous section we have chosen ψ to be single valued. Other choices which lead to a different physics are also possible. For example, one may take ψ_0 to be single valued (see, e.g., Bocchieri's and Loinger's papers in [8]). The transformation (6.1), being unitary, all observables for ψ (which in the present case is the multivalued solution of (4.1)) and ψ_0 (the single-valued solution of the free Schrödinger equation) are the same, so the magnetic field contributes nothing to the scattering and there is no AB effect. The indeterminacy of the choice of the wavefunctions merely reflects the existence of the inequivalent representations of the angular momentum in the idealised multiconnected space. The equal status of these representations and the lack of a reliable criterion† for choosing one of them have given rise to the recent excited discussion on the AB effect existence [7, 8].

On the other hand, the real experiments are performed in a simply connected space where only single-valued functions are permissible. People who deny the existence of the AB effect attribute the positive outcomes of the experiments to the particle penetration into the regions with $H \neq 0$. This may be due to either boundary effects (i.e. magnetic field leakages near the real cylindrical solenoid ends) or the finite value of the potential barrier keeping particles out of the region with $H \neq 0$. To clarify the role of these reasons, we proceed as follows. At first, we surround the infinite cylindrical solenoid with the potential barrier of finite height V_0. We estimate the contribution of different space regions to the scattering amplitude. It turns out that finite values of V_0, which permit particles to reach the $H \neq 0$ region, cannot explain the AB effect. To get rid of the boundary effects we turn to the toroidal solenoid. We surround it with the finite potential barriers of different geometrical forms. In each case the particle penetration to the region with $H \neq 0$ tends to zero as the barrier height $V_0 \to \infty$. On the other hand, the magnetic scattering amplitude tends to zero for some geometries, while for others it remains finite. Such a distinct behaviour of the magnetic scattering amplitudes could not be explained in terms of the $H \neq 0$ scattering.

8.1. Scattering on the penetrable cylindrical solenoid

Let the cylindrical solenoid of the radius R be surrounded with the cylindrical potential barrier C_0 of the finite height V_0 and of the radius $b > R$. Then, in the first Born approximation one has for the magnetic scattering amplitude

$$f_1 = \frac{1}{(2\pi i k)^{1/2}} \sum_{m \neq 0} f_{1m} \exp(\mathrm{i} m\varphi).$$

† For instance, the application of the single-valuedness and Pauli's criteria to the infinite cylindrical solenoid leads to different physical situations (there is AB scattering in the first case and no such scattering in the second one [4, 25]).

Here m is the angular momentum of the scattered particle and k is its wavenumber outside C_0. The partial amplitudes f_{1m} are:

$$f_{1m}=\mathrm{i}\pi\gamma m\left(-\frac{4}{\pi^2h_m^2}[I_{|m|}^2(k_1R)-I_{|m|-1}(k_1R)I_{|m|+1}(k_1R)]\right.$$

$$+\frac{8}{\pi^2h_m^2}\frac{1}{|m|}\sum_{s=0}^{|m|}(-1)^{|m|-s}\Delta(0,s,m)[I_s^2(k_1b)-I_s^2(k_1R)]$$

$$\left.+\frac{2}{|m|}\sum_{s=0}^{|m|}\Delta(0,s,m)[J_s(kb)-r_mH_s^{(1)}(kb)]^2\right).$$

Here k_1 is the wavenumber inside the solenoid

$$k_1=\frac{[2\mu(V_0-E)]^{1/2}}{\hbar}$$

J_s, $H_s^{(1)}$ and I_s mean the Bessell, Hankel and modified Bessel functions:

$$r_m=j_m/h_m \qquad j_m=kbI_{|m|}(k_1b)\dot{J}_{|m|}(kb)-k_1b\dot{I}_{|m|}(k_1b)H_{|m|}^{(1)}(kb)$$

$$h_m=kbI_{|m|}(k_1b)H_{|m|}^{(1)}(kb)-k_1b\dot{I}_{|m|}(k_1b)H_{|m|}^{(1)}(kb).$$

The dot over the Bessel functions means derivation with respect to their arguments. At last

$$\Delta(0,s,m)=(1+\delta_{0s})^{-1}(1+\delta_{|m|,s})^{-1} \qquad \gamma=\frac{e\phi}{hc}$$

ϕ is the magnetic field flux through the solenoid's cross section. The first line in this equation is due to the magnetic field inside the solenoid, the second one comes from the magnetic field outside the solenoid but inside C_0. Finally, the third line originates from the magnetic field outside C_0. When V_0 grows the contribution of those space regions which are inside C_0 (and hence those, for which $H\neq 0$) continuously decreases. For $V_0=\infty$ one obtains the usual AB scattering amplitude for the impenetrable solenoid:

$$f_m^{\mathrm{AB}}=2\mathrm{i}\pi\gamma\frac{m}{|m|}\sum_{l=0}^{|m|}[J_l(kb)-s_mH_l^{(1)}(kb)]^2\Delta(0,l,m)$$

$$(s_m=J_{|m|}(kb)/H_{|m|}^{(1)}(kb)).$$

We conclude: for large values of the potential barrier the contribution of the $H\neq 0$ regions to the magnetic scattering amplitude is negligible. Thus, positive outcomes of the experiments testing the AB effect cannot be attributed to the finite values of the barrier's height.

8.2. Scattering on the penetrable toroidal solenoid

To get rid of the boundary effects we surround the toroidal solenoid with the finite barriers of different geometrical forms. We demonstrate the method used surrounding first the toroidal solenoid $(\rho-d)^2+z^2=R^2$ with the spherical potential barrier S_0 of the radius $R_0>d+R(V=V_0$ inside S_0, and 0 outside it, see figure 6). Then, one obtains for the magnetic scattering amplitude in the first Born approximation

$$f_1=\sum f_lP_l(\cos\theta)$$

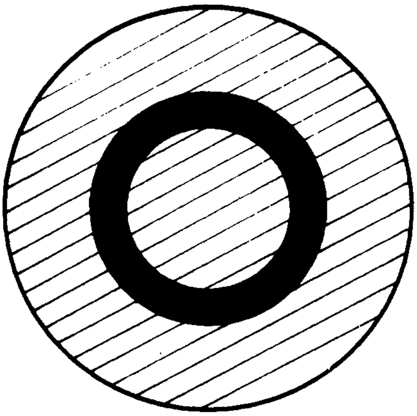

Figure 6. The toroidal solenoid is embedded into the spherical potential barrier of height V_0. The magnetic scattering amplitude tends to zero as $V_0 \to \infty$.

$$f_l = \frac{e}{4\hbar c}\left(\frac{2}{\pi i k}\right)^{1/2}(-i)^l \int dV' \frac{1}{\sqrt{r'}} R_l(r') P_l(\cos\theta') \boldsymbol{A}\boldsymbol{\nabla}\psi_0 .$$

Here ψ_0 is the wavefunction in the absence of the magnetic field, R_l is its radial part

$$\psi_0 = \left(\frac{\pi}{2kr}\right)^{1/2} \sum i^l\,(2l+1) R_l(r) P_l(\cos\theta).$$

Now we separate volume integration in f_l into two parts corresponding to the interior and exterior of the solenoid. In the latter we set $A = \operatorname{grad}\chi$, integrate twice in parts and present as a volume integral from divergence. Applying the Gauss theorem we transform this integral into the surface ones: over the solenoid surface, over the discontinuities of the χ function and over the surface of the sphere S_1 of large radius R_1. Due to the asymptotic behaviour of the χ function ($\sim r^{-2}$) the integral over the S_1 surface vanishes in the limit $R_1 \to \infty$. So, there remains integration over the nearest vicinity of the solenoid which lies completely inside S_0 (note that singularities of the χ function are also inside S_0). As both R_l and ψ_0 tend to zero inside S_0 as $V_0 \to \infty$, so does the magnetic scattering amplitude f_1. More generally, the magnetic field contribution to the scattering amplitude tends to zero as the barrier's height V_0 increases if the space regions with $H \neq 0$ as well as the singularities of the χ function are inside the barrier. This is illustrated also in figure 2 where a toroidal solenoid is surrounded with the cylindrical potential barrier or in figure 3 where it is embedded into one arm of the toroidal barrier. On the other hand, if the space regions with $H \neq 0$ are inside the potential barrier but singularities of the χ function are completely or partly outside it, then the contribution of the magnetic field to the scattering amplitude remains finite as the barrier's height increases. This is illustrated in figure 1 where the toroidal solenoid is surrounded by the toroidal potential barrier of the height V_0. For large values of V_0 the magnetic scattering amplitude tends to (4.9). Thus, in some cases (toroidal solenoid in spherical or cylindrical potential barriers or in one arm of the toroidal potential barrier) the contribution of the magnetic field to the scattering amplitude vanishes as the barrier's height grows. In other cases (see, for example, figure 1) it tends to a finite value. Such a distinct behaviour appears strange if one interprets the AB effect experiments as scattering on space regions with $H \neq 0$. In fact, the particle penetrability into the $H \neq 0$ region is practically the same for the barriers of different geometry and in each treated case goes to zero as $V_0 \to \infty$. On the other hand, this fact has natural interpretation either in terms of the non-integrable phase factor [23] (the above contribution remains finite if after the transition to the limit $V_0 \to \infty$ there remain closed paths accessible for particles along which $\oint A_l \, dl \neq 0$) or by means of the generating function (the magnetic field contribution remains finite if the potential barrier does not completely shield the singularities of the χ function).

9. On the experiments testing the Aharonov–Bohm effect

Beautiful experiments confirming the existence of the AB effect were performed by Japanese physicists [26] who studied the electron scattering on the screened toroidal magnet. The magnetic field shielding was achieved by the Cu and superconductor layers on the solenoid surface. The magnetic field leakages and particle penetration into the $H \neq 0$ region were extremely small in this experiment. The opponents of the AB effect assert, however, that however small these things are they can simulate the AB interference shift (see, e.g., Loinger's paper in [8]). The following addendum to these experiments will probably clear up any doubts about the existence of the AB effect.

We turn again to (4.9), determining the magnetic field contribution f_1 to the total scattering amplitude f. The last one is the sum of the scattering amplitude f_0 in the absence of the magnetic field, and f_1. Looking at equation (4.9) we observe that f_1 depends only on the inner torus radius a and on the magnetic field flux ϕ. The amplitude f_0 depends only on the minor a and major b radii of the torus encompassing the solenoid. Then it is possible (for fixed a, b and ϕ) to vary the solenoid and layer parameters without changing f. This is illustrated in figure 7 where one sees two configurations entirely equivalent from the usual quantum mechanical viewpoint (in the framework of which the above formula (4.9) was obtained and which treats the AB effect as scattering on space regions with $H = 0$, $\mathbf{A} \neq 0$). On the other hand, they are quite different from the alternative viewpoint (in fact, the particle penetrability into the $H \neq 0$ region is much smaller for the left configuration than for the right one). The next fact that could be tested experimentally is the dependence of the interference shift (when the current is switched off and on) on the solenoid's orientation. It is maximal when the initial wavevector is normal to the plane on which the singularities of the χ function are situated (it coincides with the equatorial solenoid plane).

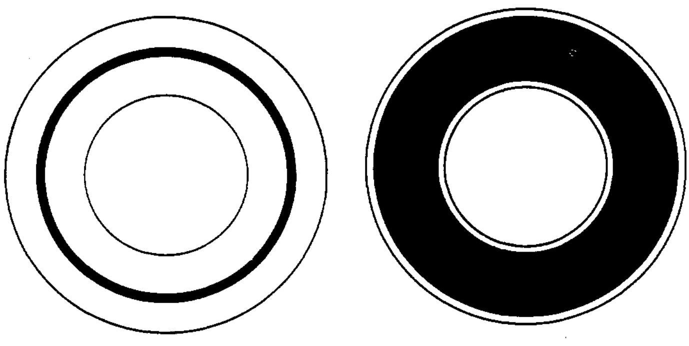

Figure 7. Two typical configurations are shown which exhibit the same quantum mechanical scattering. However, they are completely non-equivalent from the alternative viewpoint on the AB effect.

The experimental investigation of particle scattering on the toroidal solenoid embedded either into the sphere (figure 6) or one arm of the torus (figure 3) is also interesting†. The usual quantum mechanical treatment of the AB effect predicts zero shift of the interference pattern in both cases while the alternative treatment gives a finite value for the same shift.

10. The observable effects of the hidden fields in simply connected spaces

We again consider the toroidal solenoid $(\rho-d)^2+z^2=R^2$ embedded in the impenetrable sphere. The space accessible for particles is simply connected. Is it possible to detect the existence of the magnetic field by carrying experiments outside the sphere only? There are no paths accessible for particles along which $\oint A_l \, \mathrm{d}l \neq 0$. According to [23], no such experiment is possible. The following gedanken experiment seems to contradict this statement. We observe that the magnetic field inside the solenoid

$$H=\frac{\phi}{2\pi\rho}[d-(d^2-R^2)^{1/2}]^{-1}$$

results in the following increasing of the solenoid mass:

$$\Delta m=\frac{\varepsilon_{\text{magn}}}{c^2} \qquad \varepsilon_{\text{magn}}=\frac{1}{8\pi}\int H^2 \, \mathrm{d}V=\frac{1}{8\pi}\frac{\phi^2}{d-(d^2-R^2)^{1/2}}.$$

This in turn modifies the gravitational field outside the solenoid. Thus the scattering process for the particle with a finite mass looks different when the current is switched on or off. In the same way the current switching changes the parameters of the bound state orbits.

11. Conclusion

We summarise the main results obtained.

(i) The contribution of the magnetic field surrounding the toroidal solenoid to the scattering amplitude is evaluated in the first Born and high-energy approximations.

(ii) The conditions are stated when the magnetic field does not contribute to the scattering amplitude.

(iii) We study what properties the gauge transformation eliminating the vector potential in some region of space should have in order not to change the boundary conditions of the wavefunction.

(iv) Surrounding the toroidal solenoid with the penetrable potential barriers of different geometrical forms we estimate the contribution of the different space regions to the magnetic scattering amplitude. It turns out that a positive outcome of the experiments testing the AB effect could hardly be explained by the particle penetration into the $H \neq 0$ region.

(v) Experiments are proposed which probably could clear up any doubts about the existence of the AB effect.

(vi) The gedanken experiment is presented which shows that the effect of inaccessible fields could be observed even in a simply connected space.

† Due to the ambiguity of the path integral formulation in a multiconnected space the latter case is particularly important. See, e.g., [27] or papers by Ohnuki and Inomata in [6] and, especially, the subsequent discussion.

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